Ph.D. Preliminary Examination in Numerical Analysis
Department of Mathematics
New Mexico Institute of Mining and Technology
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1. This exam is four hours long. It is closed-book and cheat sheets, notes and calculators are not allowed.
2. Work out all six problems.
3. Start solution of each problem on a new page.
4. Number all of your pages.
5. Sign your name on the following line and put the total number of pages.
6. Use this sheet as a cover-sheet for your papers.

NAME: ______________________________

No. of pages:___________

Problem 1. For the initial value problem

$$\frac{dy}{dt} = f(t,y), \quad t_0 \le t \le T, \quad y(t_0) = y_0,$$

consider the following Runge-Kutta method:

$$\begin{aligned} Y_0 &= y_0 \\ Y_{n+1} &= Y_n + hf(t_n + \frac{h}{2}, Y_n + \frac{h}{2} f(t_n, Y_n)), \quad n = 0, 1, 2, ... \end{aligned}$$

where $h = t_{n+1} - t_n$. Use the Taylor expansion technique to prove that this method is second order accurate; that is, prove that the local truncation error of the scheme is of order two.

Problem 2. Consider the following theorem:

Theorem 1. *If $x_0, x_1, \cdots, x_n$ are distinct real numbers, then, for arbitrary values $y_0, y_1, \cdots, y_n$, there is unique interpolating polynomial P of degree at most n such that*

$$P(x_i) = y_i, \qquad 0 \le i \le n$$

a) Prove uniqueness of the interpolating polynomial.

b) Describe Newton's divided differences method of computing the interpolating polynomial.

Problem 3.

Let $f''(x)$ be continuous on the interval $[a,b]$, and let

$$a = x_0 < x_1 < \cdots < x_n = b$$

be a partition of $[a,b]$. Let $S(x)$ be the natural cubic spline interpolating function f at the knots $x_0, \ldots, x_n$. Prove that

$$\int_a^b (f''(x))^2 \, dx \ge \int_a^b (S''(x))^2 \, dx.$$

Hint. Let $g = f - S$. Using the integration by parts on

$$\int_a^b S''g'' \, dx = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} S''g'' \, dx,$$

telescoping series, the fact that S''' is constant on each sub-interval, the interpolation conditions, and the boundary conditions $S''(x_0) = S''(x_n) = 0$ for natural splines, prove that

$$\int_a^b S''g'' \, dx = 0.$$

Problem 4.

a) Use Newton's method to derive an algorithm for computing the 5th root of a positive real number, a.

b) Show that your iteration will converge to $\sqrt[5]{a}$ from any starting point $x_0 > 0$.

Problem 5. Let A be a positive definite matrix. Consider a descent iterative method for solving a linear system $Ax = b$ such that, given an approximation $x^{(k)}$ and a nonzero search direction $p^{(k)}$, a new approximation $x^{(k+1)}$ is computed by

$$x^{(k+1)} = x^{(k)} + \alpha_k p^{(k)}$$

for some value of α_k. Let

$$J(y) = \frac{1}{2} y^T A y - y^T b.$$

a) Describe the exact line search method for finding α_k; that is, find α_k which is the unique solution of the minimization problem

$$J(x^{(k+1)}) = \min_{\alpha \in R} J(x^{(k)} + \alpha p^{(k)}).$$

b) Let $r^{(k)} = b - Ax^{(k)}$ and $e^{(k)} = x^* - x^{(k)}$, where x^* is the exact solution of $Ax = b$, be the residual and the error vectors, respectively. Show that

$$r^{(k+1)} = r^{(k)} - \alpha_k A p^{(k)}.$$

Using this identity, prove that vector $r^{(k+1)}$ is orthogonal to both $r^{(k)}$ and $Ae^{(k)}$.

Problem 6. Consider a problem of stability of the evaluation of function f at point x. For a given absolute error h in x, the condition number of f at x can be defined as the ratio of the relative errors in $f(x)$ and x:

$$\text{cond}(f, h) = \frac{\left| \frac{f(x+h) - f(x)}{f(x)} \right|}{\left| \frac{h}{x} \right|}.$$

a) Assuming that $f'(x)$ exists, find

$$\lim_{h \to 0} \text{cond}(f, h).$$

The resulting formula is used to compute the condition number of a smooth function f.

b) Using the obtained formula, find the condition number of $f(x) = \sin x$.

c) Very large values of the condition number indicate on the large relative error in computing $f(x)$. Find the values of x for which the calculation of $\sin x$ is extremely sensitive to small relative errors in x.

d) Can a problem arise with the calculation of $f(x)$ when $|x|$ is very large, and when it is very small? Explain.