PhD Preliminary Exam in Probability & Statistics, Fall 2021

Answer all 7 questions. Each part of each question is worth 5 points. Give numerical answers whenever possible.

The exam duration is 4 hours. The exam is closed notes. The students are allowed to use a graphing calculator.

Normal, t and χ^2 tables are attached to the exam.

1. Let $X_1, X_2, \ldots, X_n$ be independent random variables with PDF

$$f(x \mid \theta) = \begin{cases} \frac{1}{\theta} x^{\frac{1-\theta}{\theta}} & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

with the parameter $\theta > 0$.

(a) Find the maximum likelihood estimator (MLE) for θ, call it $\hat{\theta}$. Calculate the estimate numerically for $n = 4$ and $X_1 = 0.10$, $X_2 = 0.22$,$X_3 = 0.54$ and $X_4 = 0.36$.

(b) Find the method of moments estimator for θ, call it $\tilde{\theta}$. Calculate the estimate numerically for $n = 4$ and $X_1 = 0.10$, $X_2 = 0.22$,$X_3 = 0.54$ and $X_4 = 0.36$.

2. Consider the following joint density for random variables X and Y:

$$f(x, y) = \begin{cases} 6xy & \text{for } 0 < x < 1, 0 < y < \sqrt{x} \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find marginal densities $f_X(x)$ and $f_Y(y)$. Are X, Y independent?

(b) Find the conditional density of X given $Y = y$.

(c) Find $\mathbb{E}\,(X \mid Y = y)$

(d) Find $Var(X \mid Y = y)$

3. Let $X_1, X_2, \ldots, X_n$ be independent random variables following Poisson distribution with the unknown mean θ. The prior distribution for θ is $Gamma(\alpha, \beta)$ with the PDF

$$\frac{\theta^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} e^{-\theta/\beta}, \quad \theta > 0, \alpha > 0, \beta > 0$$

(a) Show that the posterior distribution of θ is again a gamma distribution with parameters $\alpha^* = \alpha + \sum X_i$ and $\beta^* = \frac{\beta}{1+n\beta}$

(b) What is the Bayes estimator (under the square loss) for θ?

(c) Is the Bayes estimator for θ consistent?

4. A baseball player will go to the plate six times during a game. 20% of the time that the player goes to the plate, he gets a walk, and thus cannot get a hit. The other 80% of the time, the player gets an official "at bat". For each "at bat", there is a 30% chance of getting a hit.

(a) Use conditioning to determine the player's expected number of hits per game.

(b) Use conditioning to find the probability that the player will get no hits in a game.

5. Let $Y_1, Y_2, \ldots, Y_n$ be independent and identically distributed with probability density function given by

$$f(y) = \begin{cases} \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{y^2}{2\theta}}, & \text{for } -\infty < y < \infty,\ \theta > 0 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find the Cramer-Rao lower bound for an unbiased estimator of θ.

(b) Find the MLE $\hat{\theta}$ of θ.

(c) Is $\hat{\theta}$ an unbiased estimate of θ? Why or why not ?

(d) Find the MLE of $\ln\theta$ and justify your answer.

6. Let $X(t)$ be a pure birth process with initial value $X(0) = 1$ and the birth rate $\lambda_n = \lambda n$. Let $P_n(t) = P(X(t) = n)$. Find a system of differential equations for $P_n(t)$ and show that their solution is

$$P_n(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}, n \geq 1.$$

7. A Markov chain is defined by a random walk on the graph pictured below. From a given node, you are equally likely to go to any neighboring node.

(a) Specify the transition matrix and find the stationary distribution for this Markov chain.

(b) Find the expected time it takes, when starting from A, to visit D.