Differential Equations Prelim Spring 2014

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0.1 ODEs and OΔEs

1. Find the equilibria and their stability types for the system

$$\begin{aligned}\dot{x} =& \alpha x + y \\ \dot{y} =& x^2 + x - y^2\end{aligned}$$

2. Analyze the following system and show that it has a closed orbit.

$$\begin{aligned}\dot{p} =& q \\ \dot{q} =& p + pq - p^3\end{aligned}$$

3. Analyze the system

$$\begin{aligned}r_{n+1} =& 1 + \alpha s_n - \beta |r_n| \\ s_{n+1} =& - r_n,\end{aligned}$$

especially for the values $\alpha = 0,\ 1/2,\ 1$ and $\beta = 1, \frac{7}{4}$. What can you say about the sets of periodic orbits?

0.2 PDEs

Instructions Please show your work if you wish to be eligible for partial credit. And, don't worry too much about inequalities; whether they be $>$ or $\geq$.

1. Determine the solution of the initial value problem (uni-directional, nonlinear wave equation):

$$uu_x + u_t = 0 \quad , \quad u(x,0) = \begin{cases} 1 & , \quad -\infty < x < -3 \\ \frac{3-x}{6} & , \quad -3 \leq x < 0 \\ \frac{1-x}{4} & , \quad 0 \leq x < 1 \\ 0 & , \quad 1 \leq x < \infty \end{cases}$$

Determine when the wave breaks AND the equation of the shock(s). Summarize your solution with a *graph* in the x-t plane: $u =$ *whatever* in R_I , etc.

2. Solve Poisson's equation:

$$u_{xx} + u_{yy} = \delta(x - x_0)\delta(y - y_0)$$

in the region $R = \{(x, y)|0 < x < a,\ 0 < y < b\}$, assuming $(x_0, y_0) \in R$ subject to the boundary conditions $u(0, y) = u(a, y) = 0$, $u_y(x, 0) = u(x, b) = 0$

Give your answer in the form of a single Fourier sine series:

$$u(x, y) = \frac{2}{a}\sum_{n=1}^{\infty} A(n, y) \sin\left(\frac{n\pi x}{a}\right)$$

The function $A(n, y)$ may be written in a piecewise manner: $A(n, y) = \begin{cases} A_1(n, y) & , \quad 0 < y < y_0 \\ A_2(n, y) & , \quad y_0 < y < b \end{cases}$

3. Find the solution of the boundary value problem

$$\begin{array}{rcl} u_t = u_{xx} - \alpha u + g(t) & , & x, t > 0 \\ u(x, 0) = f(x) & , & 0 \le x < \infty \\ u_x(0, t) = 0 & , & t > 0 \end{array}$$

with $u < \infty$ as $x, t \to \infty$

Simplify the solution for the case when $f(x) = \begin{cases} 4 & , \quad 0 \le x < 2 \\ 0 & , \quad 2 \le x < \infty \end{cases}$, $\alpha = \frac{1}{4}$ and $g(t) = 1 + e^{-\alpha t}$.

Plot $u(0, t)$ for $t \in [0, 12]$.